Coupled Mode Propagation in Elastic Media

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LONG TERM GOALS

Develop an accurate and reliable range-dependent propagation model for propagation in an ocean overlying an elastic bottom based on coupled mode theory.

OBJECTIVES

The objective of this effort is to develop, test and validate a range-dependent elastic propagation code by developing a coupled-mode extension to the existing elastic normal mode codes.

APPROACH

In a variety of applications in shallow water acoustics, coupled mode theory is an attractive model. In addition to the parabolic equation technique (PE), it is the only other numerically efficient model that provides the solution of the wave equation in a range-dependent environment. However, while the parabolic equation technique for propagation in the ocean over a bottom modeled as a fluid has been hugely successful, up to date there is no PE model that can produce reliable results for propagation over an elastic bottom. It is the goal of this work to use elastic coupled mode theory to develop a range-dependent propagation model for waves in an ocean overlying an elastic bottom. To accomplish this goal, we plan to use the elastic coupled mode theory developed in [1], the acoustic version of which was successfully applied to the acoustic wedge in [2]. In this method, as in its acoustic counterpart, the range-dependent waveguide is divided into small stair-steps, and the field from one stair step to the next is propagated by the coupled mode differential equation, here referred to as the coupled mode engine. The coupled mode engine uses a marching technique similar to the one used in the PE method, except that in this case the grid spacing is determined by the width of the stair-steps, which is determined by the degree at which the water depth changes as a function of range. During this process, it computes the mode coupling matrices from the knowledge of the local modes and their depth derivatives. By combining the coupled mode engine with an elastic normal mode model, we plan to develop a model for propagation of waves in a range-dependent environment overlying an elastic bottom.

WORK COMPLETED

According to the elastic coupled mode model used in our work, the displacement vector and the stress tensor are expanded in terms of local modes with coefficients that depend on range as

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$$\mathbf{u}(\mathbf{r}) = \sum_n c_n(\mathbf{x})\mathbf{u}_n \exp(i\psi_n(\mathbf{x})),$$
$$\mathbf{T}(\mathbf{r}) = \sum_n c_n(\mathbf{x})\mathbf{T}_n \exp(i\psi_n(\mathbf{x})). \tag{1}$$

In the above, $\mathbf{u}_n$ and $\mathbf{T}_n$ are the local eigenfunctions corresponding to the components of the displacement vector and the stress tensor, respectively. Substituting the above expansion into the equations of motion,

$$\begin{aligned}
&\nabla_x \cdot \mathrm{T} = -\rho\omega^2\mathbf{u} - \hat{\mathbf{z}}\cdot\partial_z \mathrm{T}, \\
&\frac{1}{2}\left[\nabla_x\mathbf{u} + (\nabla_x\mathbf{u})^T\right] = \frac{1}{2\mu}\left[\mathrm{T} - \left(\frac{\kappa - 2/3\mu}{3\kappa}\right) tr(\mathrm{T})\right] \\
&-\frac{1}{2}\left[\hat{\mathbf{z}}\partial_z\mathbf{u} + (\partial_z\mathbf{u})\hat{\mathbf{z}}\right],
\end{aligned} \tag{2}$$

integrating them along the depth of the waveguide, and applying the boundary conditions of continuity of the normal components of the stress tensor and the displacement vector, results in a differential equation for the mode amplitudes:

$$2k_m\partial_x c_m + c_m\partial_x k_m = \sum_{n\neq m} A_{mn}c_n. \tag{3}$$

In the above equation k_m are the local eigenvalues and A_{mn} is the mode-coupling matrix, which is a function modes, their depth derivatives and the slope of the interface, evaluated on the two sides of the interface. The interested reader may refer to [1] for a mathematical expression for A_{mn}. We refer to the expression given by Equation (3) as the coupled mode engine. This equation uses the local modes and the mode-coupling matrix to compute the mode amplitudes as a function of range, which are used in Equation (1) to obtain the field in the waveguide.

To be able to test the coupled mode engine, it was sufficient to apply it to the simplest range-dependent environment. However, to be able to compute the field accurately using the above model, it is important that not only a complete set of modes are computed accurately, but also the mode number by which they are identified remains consistent as the field propagates through the waveguide. We decided to apply this model to the elastic version of the wedge problem used by Jensen and Kuperman [3]. In this problem, the ocean environment is composed of an isovelocity water layer over an isovelocity elastic bottom. For this environment the modes can be computed analytically, where it is easier to make sure that the computed modes satisfy the requirements imposed by Equation (3). The equations for the compressional and shear potentials in each layer are given in Equation (4), where z_1 is the water depth. The unknown coefficients in these equations are determined by applying the boundary conditions at the top and bottom surfaces and in the interface. The roots of the determinant of the resulting equation give the eigenvalues, k. The components of the displacement and stress tensor are computed from the potentials according to Equation (5).

$$
\begin{aligned}
&r_5^1 = A_1 \sin(k_1 z)\\
&r_5^2 = A_2 \sin(k_2(z-z_1)) + B_2 \cos(k_2(z-z_1))\\
&r_6^2 = A_3 \sin(\gamma_2(z-z_1)) + B_3 \cos(\gamma_2(z-z_1))\\
&k_1 = \sqrt{k_w^2 - k^2}, \quad k_2 = \sqrt{k_p^2 - k^2}, \quad \gamma_2 = \sqrt{k_s^2 - k^2},\\
&r_5 = \phi, \;\; r_6 = \psi / ik\\
&\partial_z^2 \phi + \frac{\omega^2}{c_p^2}\phi = 0,\\
&\partial_z^2 \psi + \frac{\omega^2}{c_s^2}\psi = 0.
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
&u_x = iV \;\; = ik(r_5 - \partial_z r_6)\\
&u_z = U \;\; = \partial_z r_5 - k^2 r_6\\
&\tau_{xz} = ik\mu\left[2\partial_z r_5 - (-k_s^2 + k^2) r_6\right]\\
&\tau_{zz} = \left[-(\lambda + 2\mu)k_p^2 + 2\mu k^2\right] r_5 - 2\mu k^2 \partial_z r_6
\end{aligned} \tag{5}
$$

Inclusion of attenuation in the problem causes the sound speeds to become complex, resulting in a complex determinant. Determination of the eigenvalues in this case requires searching the complex k-plane, which is not always reliable. For this reason in ocean acoustics it is common to solve the eigenvalue problem for the real part of the eigenvalues for the case with no attenuation and then use perturbation theory to find the complex part of the eigenvalues due to attenuation [4]. This gives a real determinant, which simplifies the problem since the eigenvalues can be found by performing a one-dimensional search along the real k-axis.

This eigenvalue formulation fulfills the requirement of providing a complete set of modes for all water depths. However, as will be seen in the Results section, the use of perturbation theory to determine the complex part of the eigenvalues is not accurate when the bottom is elastic, which results in errors in the computed field values. We therefore had to include attenuation in the problem, but find a method of solving the complex eigensystem, which did not rely on searching the complex k-plane. We determined that the best way to do this was to use the finite difference technique to solve the eigenvalue equations for the two-layer problem. This method also has the advantage of being able to solve the eigensystem in a waveguide with variable sound speed and density. In the water, we use the finite difference technique to solve the depth-separated wave equation for the pressure,

$$
\partial_z^2 p_n + \left(k^2 - k_n^2\right) p_n = 0, \tag{6}
$$

and in the elastic bottom we use the same technique to solve the coupled equations for the vertical and horizontal displacement [Tromp],

$$
\partial_z\left[(\kappa + 4/3\mu)\partial_z U_n - k_n(\kappa - 2/3\mu)V_n\right] - k_n\mu\left(\partial_z V_n + k_n U_n\right) + \rho\omega^2 U_n = 0, \tag{7}
$$

$$
\partial_z\left[\mu\left(\partial_z V_n + k_n U_n\right)\right] + k_n(\kappa - 2/3\mu)\partial_z U_n - k_n^2(\kappa + 4/3\mu)V_n + \rho\omega^2 V_n = 0. \tag{8}
$$

In the above equations, $k = \omega / c_w$, $\mu = \rho c_s^2$, $\kappa = \rho(c_p^2 - 4c_s^2/3)$, where μ is the rigidity, κ is the bulk modulus, c_w is the water sound speed, ρ is the bottom density, c_p is the compressional sound speed and c_s the shear sound speed in the bottom. The pressure in the water is coupled to the displacements in the bottom by applying the boundary conditions at the water-bottom interface, which states that the pressure and the normal component of the displacement are continuous and the tangential component of stress vanishes. The finite difference technique converts the above three equations into a generalized eigenvalue problem of the form $\mathbf{A} \cdot \mathbf{x} = k_n^2 \mathbf{B} \cdot \mathbf{x}$, which can be solved using any standard linear algebra software. The solution gives the eigenvalues and eigenfunctions for the displacements in the waveguide. The other field quantities are computed from these using the equations of elasticity. When attenuation is present, this method computes the complex eigenvalues and eigenfunctions reliably and meets the requirements necessary for use in Equation (3).

To be able to validate the solution of the wave equation in a range-dependent waveguide overlying an elastic bottom using Equation (3), we also solved the same problem using the virtual source technique. This formulation is described in detail in [abawi (4)].

RESULTS

We applied the coupled mode model described in the previous section to propagation in a wedge-shaped ocean. The ocean environment is the elastic version of the environment used by Jensen and Kuperman [3], in which the waveguide is composed of a 200-meter water layer for the first 5 km and then the water depth decreases to zero linearly in the next 10 km, resulting in a wedge angle of 1.15 degrees. We simulated propagation in this environment for two cases: In the first case the bottom compressional sound speed was 3000 m/s, the shear sound speed was 1800 m/s, the compressional attenuation was 0.5 dB per wavelength and the shear attenuation was 0.25 dB per wavelength. In the second case, the compressional sound speed was 1700 m/s and the shear sound speed was 700 m/s. The bottom attenuation for this case was the same as case 1. In these simulations, we used the analytic solution described by Equations (4 and 5) to compute the modes and computed the imaginary part of the eigenvalues due to attenuation using perturbation theory. The compressional and shear pressure field for both cases are shown in Figure (1). There are three propagating modes in this problem and for case 1, where the bottom shear sound speed is larger than the water sound speed, it is clearly seen that each mode cuts off as the water depth decreases. It also demonstrates the mode-coupling phenomenon, by which modes do not suddenly cut off and transfer their energy to the next lower-order mode, but slowly transfer their energy to the entire continuous mode spectrum [JK].

The right panel in Figure (1) shows the compressional and shear pressure fields for the case when the shear sound speed is smaller than the water sound speed. As can be seen, the field characteristics are very different. The most obvious feature is the lack of a clear presence of radiating beams into the bottom. When the shear sound speed is smaller than the water sound speed, modes join the shear mode spectrum after cut-off, which is extremely lossy. We associated the absence of radiating beams into the bottom to a loss mechanism and hypothesized that it was this mechanism, which caused modes to disappear before they had a chance to show up as radiating beams.

To be able to verify the validity of this model in general, and this hypothesis in particular, we solved a similar range-dependent problem using the virtual source technique. This technique is described in detail in [abawi (4)]. The ocean environment was composed of a wedge, where the water sound depth decreases from 200 m to zero in 4 km, resulting in a wedge angle of 2.86 degrees. We chose to solve this problem using the virtual source technique since it was numerically less intensive due to its shorter range. Even though this problem is not the same as the problem we solved using the coupled-mode model, the two problems have the same qualitative features. The virtual source solution is shown in Figure (2).

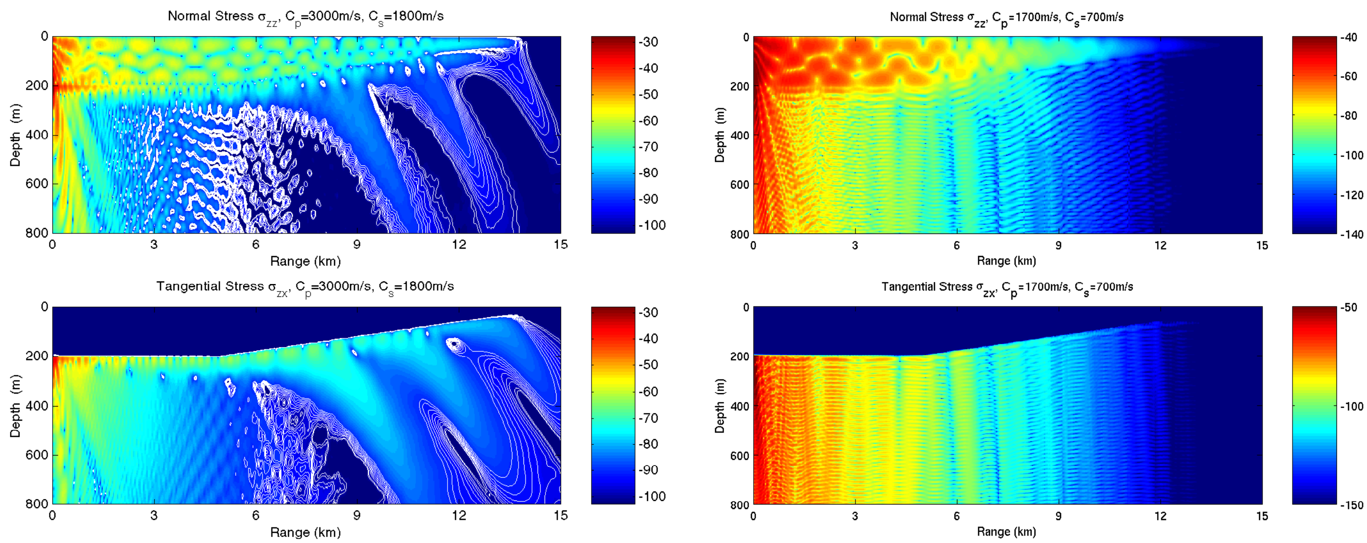

Figure 1: Propagation of waves in an elastic wedge-shaped ocean. A 25 Hz source is located at z=180 m. The field is computed using coupled-mode theory. The left panel shows the compressional and shear pressure for the case when the shear sound speed in the bottom is larger than the water sound speed. The right panel shows the same for the case when the bottom shear sound speed is smaller than the water sound speed.

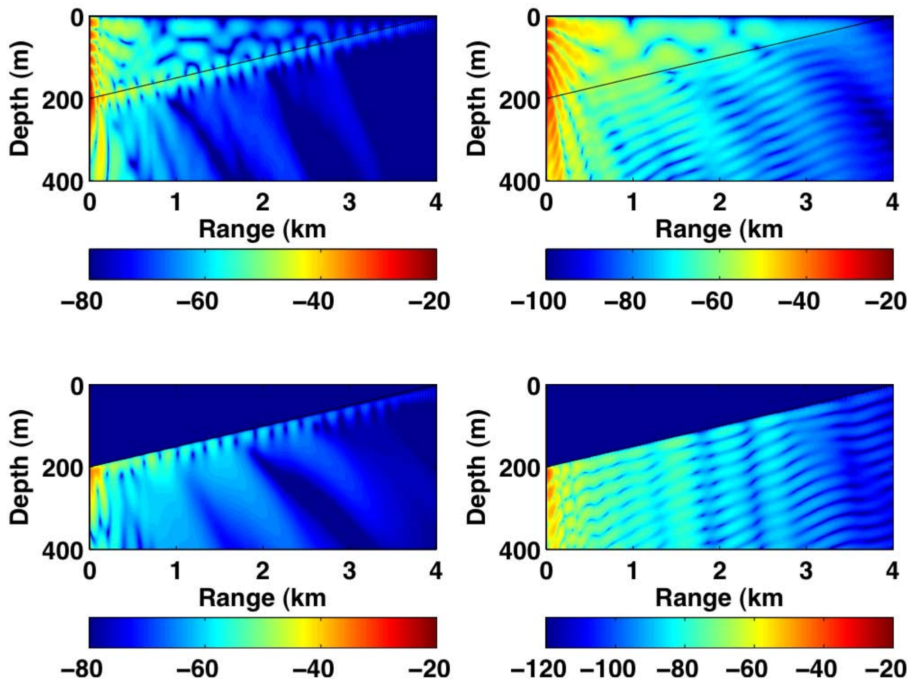

Figure 2: Propagation in an oceanic wedge using the virtual source technique. The left panel shows the compressional and shear pressure for the case when the shear sound speed in the bottom is larger than the water sound speed. The right panel shows the same for the case when the bottom shear sound speed is smaller than the water sound speed. The water and bottom sound speeds, attenuations and the source location and frequency are the same as in the case shown in Figure (1).

By comparing the results shown in Figure (1) and Figure (2), we observe that for the case when the shear sound speed is larger than the sound speed in the water, the two solutions have similar qualitative behavior. They both show modes cutting off and radiating into the bottom as beams. For the case with the shear sound speed smaller than the water sound speed, the two solutions are different, particularly for compressional pressure. While the virtual source solution shows beams radiating into the bottom during mode cut-off, this feature is absent in the coupled-mode solution. These differences between the two solutions suggested that we needed a better way of computing the elastic modes.

As a means of developing an accurate and reliable method for computing modes for a waveguide composed of multiple elastic layers with variable sound speed and density, we solved Equations (6, 7 and 8) using the finite difference method. The finite difference technique is capable of accurately and reliably computing the complex modes in a waveguide overlying an elastic bottom at any water depth. We used this technique to compute transmission loss in a waveguide composed of a water layer over an elastic bottom and compared the results with the those obtained using the wavenuber integration method, OASES {REF}. We first used the analytic solution; Equations (4 and 5) to compute the transmission loss for a waveguide overlaying an elastic bottom without attenuation. In this case the modes are real and they can be found by searching for the zeros of the eigenvalue equation on the k-axis. A comparison between our results and those obtained from OASES are shown in Figure (4).

Figure 3: Comparison of transmission loss computed using the analytic solution and the OASES solution for a waveguide composed of a 200 m water layer and 600 m elastic bottom layer for the case with no attenuation in the bottom. A 25-Hz source is located at 180 and the receiver is at 30 m. The bottom compressional sound speed is 1700 m/s and its shear sound speed is 700 m/s.

It can be seen that the analytic solution agrees perfectly with the OASES solution. We next added attenuation to the bottom and used the analytic solution to compute transmission loss. In this case we used perturbation theory to compute the imaginary part of the modes. A comparison of this result and that of OASES are shown in Figure (5). It is clear from Figure (5) that the use of perturbation theory to compute the imaginary part of the modes does not produce accurate results when the bottom is elastic. We next used the finite difference technique to compute the complex modes and compared the resulting solution with that obtained from OASES. This comparison is shown in Figure (6). We observe that the finite difference solution shows excellent agreement with the solution from OASES. This shows that this technique computes the modes correctly even when they are complex due to attenuation. This technique has all the necessary features to be used as a mode computation module in our elastic coupled mode model.

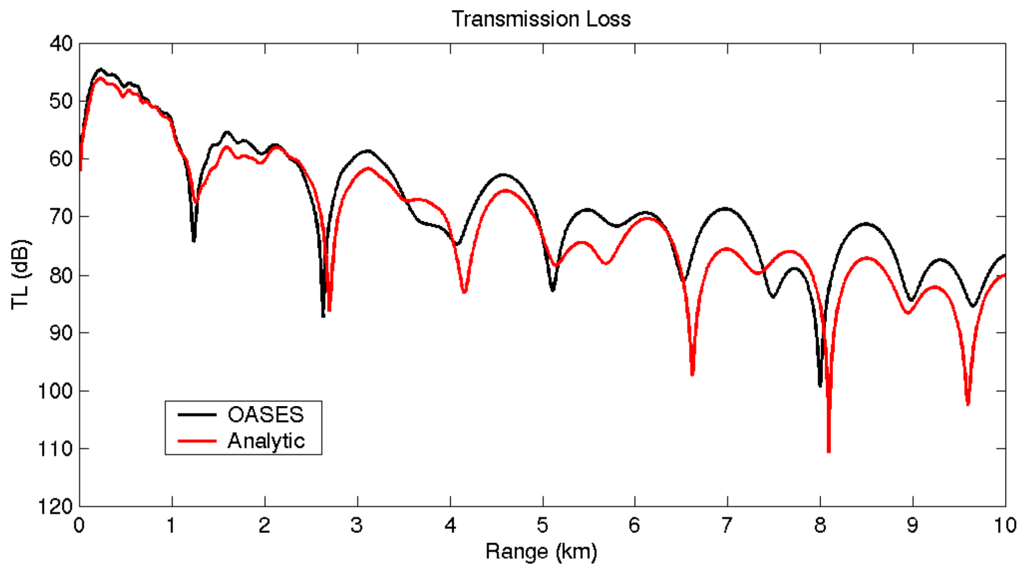

Figure 4: Comparison of transmission loss computed using the analytic solution and the OASES for a waveguide with the same parameters as in Figure (5), but with bottom attenuation. In this case the compressional bottom attenuation is 0.5 dB/wavelength and the shear attenuation is 0.25 dB/wavelength.

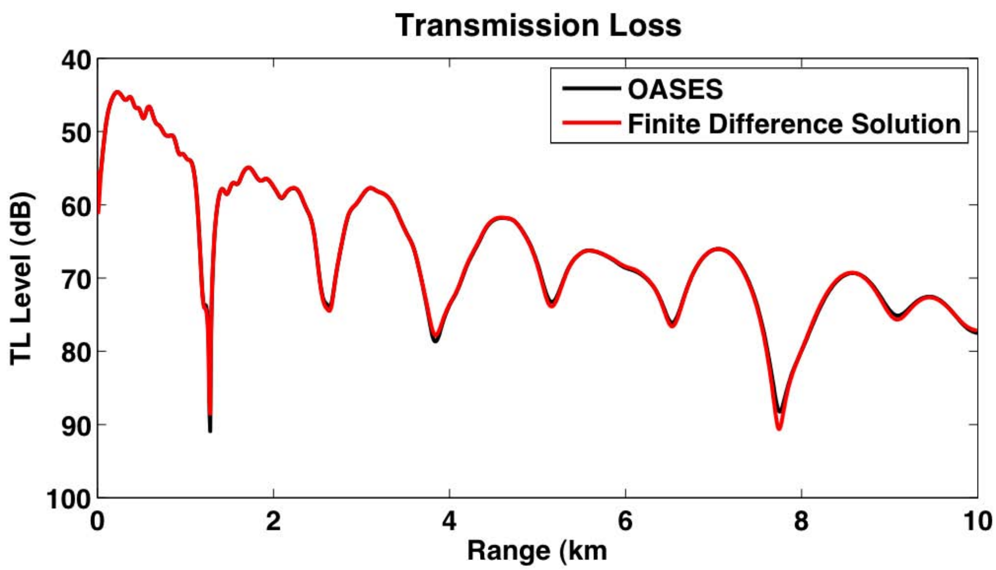

Figure 5: Comparison of transmission loss computed using the finite difference solution and the solution from OASES for a waveguide with the same parameters as the one shown in Figure (5).

Its only shortcoming at the moment is that is computationally slow, as it requires a very fine grid. We are currently working to improve this by using a more accurate finite differencing scheme and in parallel considering the use of finite element technique using high order basis functions.

IMPACT/APPLICATIONS

The finite difference technique that we developed to compute elastic modes for our elastic coupled mode model is capable of accurately computing modes for any combination of fluid and elastic layers. It therefore can be used by any normal mode or coupled mode propagation model to produce benchmark-quality solutions of the wave equation in elastic media.

TRANSITIONS

The primary transition for the technology developed under this project is the delivery of an accurate and reliable model for propagation of waves in a range-dependent ocean overlying an elastic bottom.

RELATED PROJECTS

This project benefited from another project titled "Propagation-invariant classification", also sponsored by ONR. Under this project we developed the virtual source technology, which we applied to propagation in an elastic wedge to produce a benchmark solution.

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5. Ahmad T. Abawi and Michael B. Porter, "Propagation in a wedge using the virtual source technique". J. Acoust. Soc. Am., 121 (5), May 2007.

PUBLICATIONS AND AWARDS

1. Ahmad T. Abawi and Michael B. Porter, "Propagation in a wedge using the virtual source technique". J. Acoust. Soc. Am., 121, 1374, (2007).
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